M.Sc. 2nd Semester (2020)
Measure Theory and Functional Analysis

Theorem: If G_1 & G_2 are any two open sets in $[a,b]$ then,

$$m(G_1)+m(G_2) = m(G_1 \cup G_2) + m(G_1 \cap G_2)$$

i.e. $$m(G_1)+m(G_2) = m(G_1 + G_2) + m(G_1 G_2)$$

Proof: Here $m(G_1)$ = length of interval in G_1.
The length of the interval which is common in G_1 & G_2 is equal to the length of the interval in $G_1 \cap G_2 = G_1 G_2$. On the other hand the lengths of common intervals in $m(G_1)+m(G_2)$ occurs twice. consequently

$$m(G_1)+m(G_2) = m(G_1+G_2) + m(G_1 G_2)$$

Example (i) If E_1 & E_2 are measurable subsets of $[a,b]$, prove that $E_1 \cup E_2$, $E_1 \cap E_2$ & $E_1 - E_2$ are measurable & show:-

i (a) $m(E_1)+m(E_2) = m(E_1 \cap E_2) + m(E_1 \cup E_2)$

(b) $m(E_1 - E_2) = m(E_1) - m(E_2)$ if $E_2 \subset E_1$.

Solu^n^ (a) E_1 & E_2 are measurable subsets of $[a,b]$

$\therefore\ m_e(E_1) = m_i(E_1) = m(E_1)$

also, $m_e(E_2) = m_i(E_2) = m(E_2)$ —— (*)

$\because\ m_e(E_1) + m_e(E_2) \geq m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$ — (1)

& $m_i(E_1) + m_i(E_2) \leq m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$ —— (2)

$\therefore$ Writing eq^n^ (1) & (2), with the help of (*), we get:-

$m(E_1) + m(E_2) \geq m(E_1 \cup E_2) + m(E_1 \cap E_2)$. — (3)

also, $m(E_1) + m(E_2) \leq m(E_1 \cup E_2) + m(E_1 \cap E_2)$ — (4)

for finite union & intersection of measurable sets are measurable, so that $E_1 \cup E_2$ & $E_1 \cap E_2$ both are measurable & so:-

$m_e(E_1 \cup E_2) = m_i(E_1 \cup E_2) = m(E_1 \cap E_2)$.

$m_e(E_1 \cap E_2) = m_i(E_1 \cap E_2) = m(E_1 \cap E_2)$

on combining (3) & (4) we get :-

$$m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2),$$

Which is the required result

(ii.) Since, difference of measurable sets is measurable.

$\therefore$ $E_1 - E_2$ is a measurable set.

$$(E_1 - E_2) \cap E_2 = \phi$$

$\therefore$ $m[(E_1 - E_2) \cup E_2] = m(E_1 - E_2) + m(E_2)$.

or, $\quad m(E_1) = m(E_1 - E_2) + m(E_2)$

or, $\quad m(E_1) - m(E_2) = m(E_1 - E_2)$.

Eg:2. The difference of two measurable set is measurable.

Soln: Let E_1 & E_2 be measurable sets

To prove that $E_1 - E_2$ is a measurable set.

$$E_1 - E_2 = E_1 \cap E_2'.$$

E_2 is measurable $\Rightarrow$ E_2' is measurable

$\Rightarrow$ $E_1 \cap E_2'$ is measurable

$\Rightarrow$ ~~$E_1 - E$~~ $E_1 - E_2$ is measurable.

Hence, proved.